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LETTER TO THE EDITOR

Three-dimensional analytical periodic solutions of the Laplace equation

D Ouroushev

University of Sofia, Faculty of Physics, Department of Solid State Physics, Boulevard A Ivanov 5, Sofia 1126, Bulgaria

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Abstract. A method is proposed, by which three-dimensional periodic analytical solutions of the Laplace equation can be found. The solutions obtained describe the electrostatic potential in a three-dimensional space lattice of point charges with a certain symmetry.

The problem of finding the three-dimensional solution of the Laplace equation

$$\Delta\psi = 0 \tag{1}$$

is usually solved by separating the variables. By this method a general solution can be obtained, which has the form (Morse *et al* 1953)

$$\psi = \mathrm{e}^{\pm irx}\,\mathrm{e}^{\pm isy}\,\mathrm{e}^{\pm tz}, \qquad t^2 = r^2 + s^2. \tag{2}$$

As can be seen from (2) this solution is periodic in the x and y directions and exponentially decreasing or increasing in the z direction. It should be mentioned here that the correlation between the constants r, s and t is always such that the coefficient of z is real. Consequently the method of separation of the variables cannot be used to obtain a solution periodic in three dimensions.

Here a method will be proposed by which a three-dimensional periodic solution of the Laplace equation can be found. Let us make the following substitution in equation (1)

$$\psi = 4q\,\mathrm{Arth}[u_1(x)v_1(y)w_1(z) + u_2(x)v_2(y)w_2(z)] \tag{3}$$

where u_i, v_i, w_i $(i = 1, 2)$ are Jacobi elliptic functions which satisfy the following non-linear ordinary differential equations (Janke *et al* 1960)

$$\begin{aligned}(\mathrm{d}u_i/\mathrm{d}x)^2 &= A_i^x u_i^4 + B_i^x u_i^2 + C_i^x\\ (\mathrm{d}v_i/\mathrm{d}y)^2 &= A_i^y v_i^4 + B_i^y v_i^2 + C_i^y \qquad i = 1, 2\\ (\mathrm{d}w_i/\mathrm{d}z)^2 &= A_i^z w_i^4 + B_i^z w_i^2 + C_i^z.\end{aligned} \tag{4}$$

Substituting (4) into equation (1) we obtain

$$\begin{aligned}&(A_1^x u_1^2 + A_1^y v_1^2 + A_1^z w_1^2)2\alpha + (A_2^x u_2^2 + A_2^y v_2^2 + A_2^z w_2^2)2\beta\\ &\quad - 2(A_1^x u_1^2 + A_1^y v_1^2 + A_1^z w_1^2)^2(\alpha\beta^2 + \beta^2\alpha) + (B_1^x + B_1^y + B_1^z)(\alpha + \alpha^3 + \alpha\beta^2)\\ &\quad + (B_2^x + B_2^y + B_2^z)(\beta + \beta^3 + \beta^2\alpha) + 2(\alpha + \beta)\end{aligned}$$

$$\times(C_1^x v_1^2 w_1^2 + C_2^x v_2^2 w_2^2 + C_1^y u_1^2 w_1^2 + C_2^y u_2^2 w_2^2 + C_1^z u_1^2 v_1^2 + C_2^z u_2^2 v_2^2)$$
$$+4(\alpha+\beta)(u_1' u_2' v_1 w_1 v_2 w_2 + v_1' v_2' u_1 u_2 w_1 w_2 + w_1' w_2' u_1 u_2 v_1 v_2) = 0. \tag{5}$$

Here the prime denotes a derivative by the corresponding variable; α and β are correspondingly

$$\alpha = u_1 v_1 w_1 \qquad \beta = u_2 v_2 w_2. \tag{6}$$

Equation (5) can be further simplified by using some concrete properties of the Jacobi elliptic functions. Let us assume that $u_i v_i w_i$ are some of the three main Jacobi elliptic functions sn(x, k), cn(x, k), dn(x, k) (Janke *et al* 1960). In this case

$$\begin{aligned} u_1' u_2' &= u_1 u_2 (D_1^x + D_2^x) + A_1^x u_1^2 u_1 u_2 + A_2^x u_2^2 u_1 u_2 \\ v_1' v_2' &= v_1 v_2 (D_1^y + D_2^y) + A_1^y v_1^2 v_1 v_2 + A_2^y v_2^2 v_1 v_2 \\ w_1' w_2' &= w_1 w_2 (D_1^z + D_2^z) + A_1^z w_1^2 w_1 w_2 + A_2^z w_2^2 w_1 w_2. \end{aligned} \tag{7}$$

Of course the relations (7) are valid if the pairs of functions $u_1 u_2$; $v_1 v_2$; $w_1 w_2$ depend on the same modal k_x, k_y, k_z respectively. In (7) D_1^{xyz}, D_2^{xyz} are constants, which can be determined by making a concrete choice for the functions $u_i v_i w_i$; they can be expressed by the coefficients B_i^{xyz}.

It must be underlined that for every combination of main elliptic functions (sn cn; sn dn; cn dn) relations (7) are valid.

Using (7) equation (5) can be simplified:

$$\begin{aligned} &(B_1^x + B_1^y + B_1^z)(\alpha + \alpha^3) + (B_2^x + B_2^y + B_2^z)(\beta + \beta^3) \\ &\quad + [2(D_1^x + D_2^x) + 2(D_1^y + D_2^y) + 2(D_1^z + D_2^z) - B_1^x - B_1^y - B_1^z]\alpha\beta^2 \\ &\quad + [2(D_1^x + D_1^y) + 2(D_1^y + D_2^y) + 2(D_1^z + D_2^z) - B_2^x - B_2^y - B_2^z]\beta^2\alpha \\ &\quad + [A_1^x u_1^2 + A_1^y v_1^2 + A_1^z w_1^2 + C_1^x v_1^2 w_1^2 + C_2^x v_2^2 w_2^2 + C_1^y u_1^2 w_1^2 \\ &\quad + C_2^y u_2^2 w_2^2 + C_1^z u_1^2 v_1^2 + C_2^z u_2^2 w_2^2]2\alpha + [A_2^x u_2^2 + A_2^y v_2^2 \\ &\quad + A_2^z w_2^2 + C_1^x v_1^2 w_1^2 + C_2^x v_2^2 w_2^2 + C_1^y u_1^2 w_1^2 + C_2^y u_2^2 w_2^2 \\ &\quad + C_1^z u_1^2 v_1^2 + C_2^z u_2^2 v_2^2]2\beta = 0. \end{aligned} \tag{8}$$

Consequently the problem of solving the three-dimensional Laplace equation is reduced to that of solving a system of algebraic equations. The first three equations of this system, as follows from (8), are

$$\begin{aligned} &B_1^x + B_1^y + B_1^z = 0 \\ &B_2^x + B_2^y + B_2^z = 0 \\ &D_1^x + D_2^x + D_1^y + D_2^y + D_1^z + D_2^z = 0. \end{aligned} \tag{9}$$

Moreover, the coefficients of 2α and 2β in (8) must be equal to zero:

$$\begin{aligned} &A_1^x u_1^2 + A_1^y v_1^2 + A_1^z w_1^2 + C_1^x v_1^2 w_1^2 + C_2^x v_2^2 w_2^2 \\ &\quad + C_1^y u_1^2 w_1^2 + C_2^y u_2^2 w_2^2 + C_1^z u_1^2 v_1^2 + C_2^z u_2^2 v_2^2 = 0 \\ &A_2^x u_2^2 + A_2^y v_2^2 + A_2^z w_2^2 + C_1^x v_1^2 w_1^2 + C_2^x v_2^2 w_2^2 \\ &\quad + C_1^y u_1^2 w_1^2 + C_2^y u_2^2 w_2^2 + C_1^z u_1^2 v_1^2 + C_2^z u_2^2 v_2^2 = 0. \end{aligned} \tag{10}$$

Equations (10) lead to a further ten relations between the coefficients $A_i^x A_i^y A_i^z C_i^x C_i^y C_i^z$ $(i=1,2)$. The exact form of these equations can be given after a concrete choice of the functions $u_i v_i w_i$. Solving the system of algebraic equations determined by (9) and (10) we can in principle find a three-dimensional periodic solution of the Laplace equation, expressed in Jacobi elliptic functions.

Let us now do a concrete choice for the functions $u_i v_i w_i$ setting

$$\begin{aligned} u_1 &= a_1^x \operatorname{cn}(lx, k_x) & u_2 &= a_2^x \operatorname{sn}(lx, k_x) \\ v_1 &= a_1^y \operatorname{cn}(my, k_y) & v_2 &= a_2^y \operatorname{sn}(my, k_y) \\ w_1 &= a_1^z \operatorname{cn}(nz, k_z) & w_2 &= a_2^z \operatorname{dn}(nz, k_z). \end{aligned} \tag{11}$$

As can be seen from (11) 18 indeterminated constants $A_i^{xyz} B_i^{xyz} C_i^{xyz}$ reduce to 12 due to the fact, that for every pair of functions u_i, v_i, w_i $(i=1,2)$ the coefficients before xyz, as the modals k_x, k_y, k_z must be the same, because in (9) and (10) these functions must be combinate.

Due to the fact that the function $u_i v_i w_i$ occurs in the solution only in the combinations $u_1 v_1 w_1$, $u_2 v_2 w_2$, the coefficients a_i^x, a_i^y, a_i^z, a_2^x, a_2^y, a_2^z can be reduced to two:

$$A = a_1^x a_1^y a_1^z \qquad B = a_2^x a_2^y a_2^z. \tag{12}$$

In this case the system of equations (9) and (10) reduces to eight equations for the eight coefficients A, B, k_x, k_y, k_z, l, m, n. Solving this system we obtain

$$A = B = 1 \qquad l^2 + m^2 = n^2 \qquad k_x^2 = k_y^2 = 1 - k_z^2. \tag{13}$$

Consequently the function

$$\psi = 4q \operatorname{Arth}[\operatorname{cn}(lx, k)\operatorname{cn}(my, k)\operatorname{cn}(nz, k') + \operatorname{sn}(lx, k)\operatorname{sn}(my, k)\operatorname{dn}(nz, k')] \tag{14}$$

where $k^2 = 1 - k'^2$, is a three-dimensional periodic solution of the Laplace equation. The solution obtained is periodic in the x, y and z direction with respective periods

$$T_x = 4K(k)/l \qquad T_y = 4K(k)/m \qquad T_z = 4K(k')/n \tag{15}$$

where $K(k)$ and $K(k')$ are the full elliptic integrals of the first kind determining the periods of the elliptic functions.

It must be mentioned that the obtained function (14) possesses singularities at the points in which the argument of the function Arth is equal to ± 1. These singularities can be interpreted according to the physical meaning of the Laplace equation. This equation describes the electrostatic potential in a system of charges in the areas where the space charge density is zero.

The presence of singularities in the solution of the Laplace equation is usually connected with the existence of a point charge in these points (Jackson 1962). Setting the constant q in (3) equal to the absolute value of these point charges it can be said that a solution of this type describes the electrostatic potential in a system of point charges distributed periodically and forming a space lattice.

For the concrete solution (14) this space lattice is given in figure 1. As can be seen from the figure, this is a space lattice from the rhomboid system with C primitive cell.

Consequently the solution obtained is an analytical expression for the electrostatic potential in the crystal-like structure of point charges with the aforementioned symmetry.

Here it must be mentioned that the proposed method gives us a possibility of finding solutions with other symmetry, which are also expressed in Jacobi elliptic

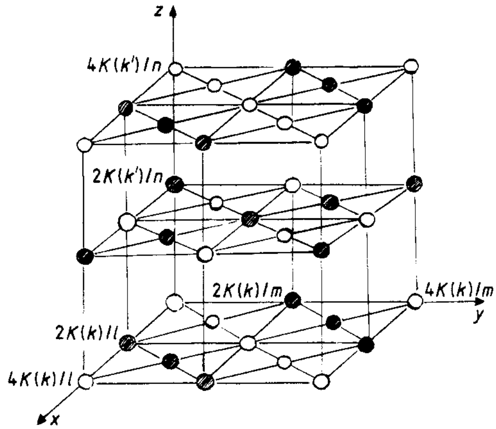

Figure 1. The space lattice of point charges described by the solution (14) of the Laplace equation. Open circles, positive point charges; full circles, negative point charges.

functions. Consequently the proposed method makes it possible to find an analytical expression for the electrostatic potential in a three-dimensional periodic structure of point charges or we obtain an analytical expression for the crystal field in a certain type of ionic crystal with corresponding symmetry.

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